Playing with Burgers's equation

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Abstract

The 1D Burgers equation is used as a toy model to mimick the resulting behaviour of numerical schemes when replacing a conservation law by a form which is equivalent for smooth solutions, such as the total energy by the internal energy balance in the Euler equations. If the initial Burgers equation is replaced by a balance equation for one of its entropies (the square of the unknown) and discretized by a standard scheme, the numerical solution converges, as expected, to a function which is not a weak solution to the initial problem. However, if we first add to Burgers' equation a diffusion term scaled by a small positive parameter ϵ before deriving the entropy balance (this yields a non conservative diffusion term in the resulting equation), and then choose ϵ and the discretization parameters adequately and let them tend to zero, we observe that we recover a convergence to the correct solution.

Key words : Burgers equation, compressible flows, Euler equations, finite volumes.
MSC2010: 65M08,76N99

1 Introduction

Computer codes developed for the simulation of inviscid and non heat-conducting compressible flows are in general based on the conservative form of the Euler equations, which read in the one-dimensional case:

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{1a}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = 0, \tag{1b}$$

$$\partial_t E + \partial_x\big((E+p)u\big) = 0, \tag{1c}$$

where t stands for the time, ρ, u and p are the density, velocity and pressure in the flow, and E stands for the total energy:

$$E = \rho\,(\frac{1}{2}u^2 + e),$$

with e the internal energy. This system must be complemented by an equation of state, giving for instance the pressure as a function of the density and the internal energy $p = \wp(\rho, e)$.

For physical reasons, the density and internal energy must be non-negative (in usual applications, positive). In addition, for the continuous problem as well as, at the discrete level, for a wide range of schemes (the so-called conservative schemes), the non-negativity of these variables allows a (weak) control on the solution; assuming that ρ and E are known on the parts of the boundary where the flow is entering the computational domain, Equations (1a) and (1c) indeed yield an $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))$-estimate (with $\Omega \times (0,T)$ the space-time domain of computation) for the density and the total energy respectively. The positivity of the density at the discrete level is easily obtained from a convenient discretization of (1a). The positivity of the internal energy does not seem easily obtained other than by replacing Equation (1c) by a balance equation for the internal energy in the discrete problem; this balance equation is formally derived (*i.e.* supposing that the solution is regular) from (1b) and (1c) and reads:

$$\partial_t(\rho e) + \partial_x(\rho e u) + p\partial_x u = 0. \tag{2}$$

In this relation, the discrete convection operator may be built so as to respect the positivity of e: provided that the equation of state is such that for any value of ρ, p vanishes for $e = 0$, testing the discrete counterpart of (2) by the negative part of e proves $e \geq 0$ (see [5] for the initial paper, [2, Appendix B] for another proof suitable in this context, and [4] in the framework of the compressible Navier-Stokes equations).

Instead of Equation (1c), one may also prefer to use a conservation equation for the physical entropy s, because this equation (derived for regular solutions) is a simple transport equation:

$$\partial_t(\rho s) + \partial_x(\rho s u) = 0. \tag{3}$$

Let us then consider that, for computational efficiency or robustess reasons, (2) or (3) are prefered to (1c). Since both (2) and (3) are derived from (1c) assuming a regular solution, there is no reason for their discretization to yield the correct weak solutions in the presence of shocks. Nevertheless, we may reasonably expect to recover the correct shock solutions if we use the following strategy:

(i) regularize the problem by adding a small diffusion term,

(ii) derive the counterpart of (2) or (3) taking into account the diffusion terms,

(iii) solve these equations,

(iv) let ϵ tend to zero.

Of course, step (*iii*) is performed numerically, and convergence is monitored by the space and time discretization steps h and k; the question which arises is then to find a convenient way to let ϵ and the numerical parameters h and k tend to zero. The aim of this paper is to perform numerical experiments in order to investigate this issue on a toy problem, namely the inviscid Burgers equation. Note that we only consider explicit schemes in this study.

2 The equations and the numerical schemes

The inviscid Burgers equation reads:

$$\partial_t u + \partial_x(u^2) = 0, \qquad \text{for } x \in \mathbb{R},\ t \in (0,T), \tag{4}$$

which we complement with the initial condition:

$$u(x,0) = u_0(x), \qquad \text{for } x \in \mathbb{R}. \tag{5}$$

Following the above mentioned strategy (items (i)-(iv)), we first add to (4) a viscous term, to obtain: $\partial_t u + \partial_x(u^2) - \epsilon\partial_{xx}u = 0$. Now, multiplying this relation by $2u$ yields the following perturbed equation:

$$\partial_t u^2 + \frac{4}{3}\,\partial_x u^3 - 2u\epsilon\partial_{xx}u = 0. \tag{6}$$

For $\varepsilon = 0$, we get the following "Burgers square entropy" equation:

$$\partial_t u^2 + \frac{4}{3}\,\partial_x u^3 = 0. \tag{7}$$

which also reads, setting $v = u^2$:

$$\partial_t v + \frac{4}{3}\partial_x(v^{\frac{3}{2}}) = 0. \tag{8}$$

We consider the following initial data, chosen such that the entropy solution of (4)-(5) contains a discontinuity:

$$u_0(x) = \begin{cases} 10, & x \le -0.25 \\ 1, & x > -0.25 \end{cases}. \tag{9}$$

It is well known that for such an initial condition, the entropy weak solutions of equations (4) and (7) differ. Let us then turn to their numerical approximations. Since the chosen initial data (9) is positive, the celebrated Godunov scheme reduces for both equations to the classical upwind scheme, thanks to the fact that the upwind scheme preserves (for these equations) the sign of the solution; it is well known that it leads to an approximate solution which converges, under a so called CFL condition, to the exact solution as the discretization parameters go to zero [1] (note that this is not the case for the centred finite volume scheme, although it is conservative). For the sake of simplicity, we consider constant time and space steps h and k. For $i \in \mathbb{Z}$, we set $x_i = ih$ and for $n \in \{0, \dots, M\}$, with $(M-1)k < T \le Mk$, we set $t_n = nk$. The discrete unknowns are the real numbers $u_i^{(n)}$, with $i \in \mathbb{Z}$ and $n \in \{0, \dots, M\}$. The values $u_i^{(0)}$ are obtained with the initial condition:

$$u_i^{(0)} = \frac{1}{h}\int_{x_i-\frac{h}{2}}^{x_i+\frac{h}{2}} u_0(x)dx. \tag{10}$$

Since the discrete solution is positive, the upwind scheme for Equation (4) reads:

$$u_i^{(n)} = u_i^{(n-1)} + \frac{k}{h}\left[\left(u_{i-1}^{(n-1)}\right)^2 - \left(u_i^{(n-1)}\right)^2\right]. \tag{11}$$

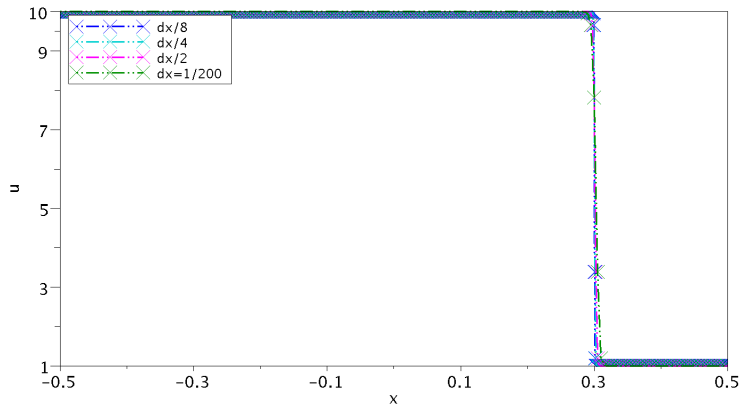
For this particular problem and scheme, the maximum value for the solution is reached at the initial time step so that the CFL number is the number G such that:

$$k = G\frac{h}{\max\{2s,\ s \in [1,10]\}} = G\frac{h}{20}. \tag{12}$$

Similarly, the upwind scheme for Equation (8) reads:

$$v_i^{(n)} = v_i^{(n-1)} + \frac{4k}{3h}\left[\left(v_{i-1}^{(n-1)}\right)^{\frac{3}{2}} - \left(v_i^{(n-1)}\right)^{\frac{3}{2}}\right], \tag{13}$$

and the CFL number is the same number G. The numerical solutions obtained with (11) for the Burgers equation (4) and with (13) for the Burgers square entropy equation (7) are depicted in



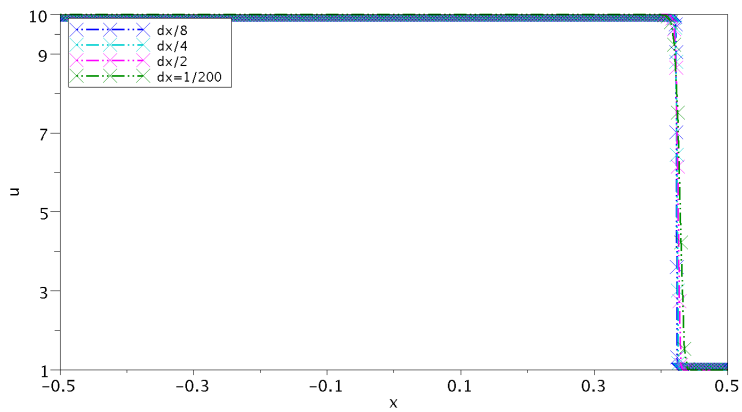


Figure 1: Upwind Scheme for (4)-(9) (top) and (7)-(9) (bottom) with different mesh sizes, $CFL = 1$.

Figure 1. Both are obtained with CFL equal to 1, for $T = 1/20$ and with various values of N, starting from $N = 200$ and multiplying successively by two the number of cells up to $N = 1600$. As expected, the upwind scheme (13) yields a numerical solution which converges (as the discretization parameters go to zero and under a CFL condition) to a weak solution of (7) (and even to its entropy solution), which is not a weak solution of (4), since the Rankine-Hugoniot conditions differ. At time $T = 1/20$, the shock for the solution of (4) is located at $x = 0.3$, while the shock of the solution of (7) is located at $x > 0.4$.

Remark 1 (Link with a non-conservative diffusion term) For the Burgers equation (4), upwinding may be seen as adding a diffusion, namely discretizing (since $u > 0$):

$$\partial_t u + \partial_x(u^2) - \partial_x((hu - 2ku^2)\partial_x u) = 0,\ x \in \mathbb{R},\ t \in (0, T).$$

Note that one has $hu - 2ku^2 \geq 0$ thanks to the CFL condition. For the Burgers square entropy equation (7), upwinding may be seen, formally, as solving the following parabolic equation (since $u > 0$):

$$\partial_t u^2 + \frac{4}{3}\partial_x(u^3) - \partial_x((2hu^2 - 4ku^3)\partial_x u) = 0.$$

This equation is equivalent to the following parabolic perturbation of the Burgers equation:

$$\partial_t u + \partial_x(u^2) - \frac{1}{u}\partial_x((hu^2 - 2ku^3)\partial_x u) = 0.$$

The third term at the left-hand side may be seen as a numerical diffusion (thanks to the CFL condition) which is not in a conservative form, because of the factor $1/u$. The above numerical results show that such a non conservative diffusion may lead to wrong discontinuities.

3 Numerical solution of the perturbed equation

We then discretize the perturbed equation (6) with $\epsilon = \epsilon_0\, h^\alpha$, where $\epsilon_0 > 0$ and $\alpha > 0$ are fixed. Note that, setting $v = u^2$, (6) can also be recast as:

$$\partial_t v + \frac{4}{3}\partial_x(v^{\frac{3}{2}}) - v^{\frac{1}{2}} h^\alpha \partial_x(v^{-\frac{1}{2}}\partial_x v) = 0, \tag{14}$$

that is a nonlinear hyperbolic equation augmented with a nonlinear nonconservative diffusion term. The upwind finite volume discretization of this equation reads (in the u variable), with $u_i^{(0)}$ given by (10):

$$\begin{aligned}\left(u_i^{(n)}\right)^2 = \left(u_i^{(n-1)}\right)^2 + \frac{4k}{3h}\left[\left(u_{i-1}^{(n-1)}\right)^3 - \left(u_i^{(n-1)}\right)^3\right]& \\ + \frac{k}{h^2}\,\epsilon_0\, h^\alpha\, u_i^{(n-1)}\left[u_{i-1}^{(n-1)} - 2u_i^{(n-1)} + u_{i+1}^{(n-1)}\right].&\end{aligned} \tag{15}$$

We present in Figures 2, 3 and 4 the numerical solutions obtained with (15) for $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 2$ respectively, and for the same time $T = 1/20$, CFL=1 and meshes as in Section 2. The parameter ϵ_0 is such that $\epsilon_0 h^\alpha = 0.2$ for $N = 200$ (whatever α may be). Figure 2 shows that for $0 < \alpha < 1$, the sequence of approximate solutions given by (15) converges to a weak solution of the initial Burgers equation (4), as h and k tend to 0, under a stability condition, which, since $\alpha < 1$, becomes more stringent than a CFL condition when h tends to zero. Figure 3 shows that for $\alpha > 1$, we obtain the convergence to the solution of (7); figure 4 shows that for $\alpha = 1$, the location of the discontinuity lies in between the discontinuities of the solution to (6) and (7). These results seem to indicate that the convergence to the solution of (7) (resp. (6)) occurs when the added diffusion dominates (resp. is dominated by) the numerical one.

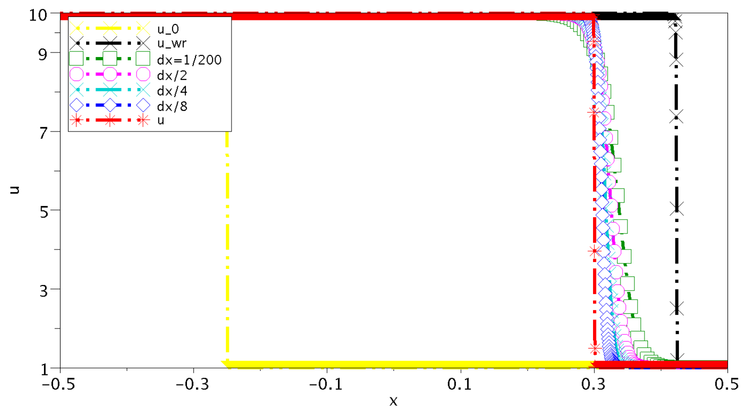


Figure 2: Upwind Scheme for (6) with non conservative diffusion term, $\alpha = 0.5$.

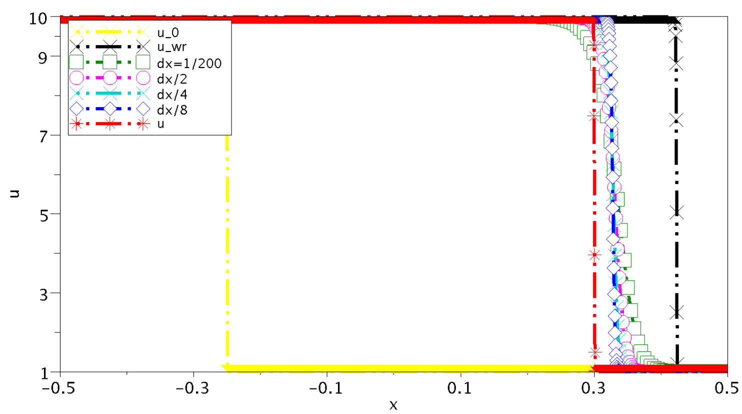


Figure 3: Upwind Scheme for (6) with non conservative diffusion term, $\alpha = 1$.

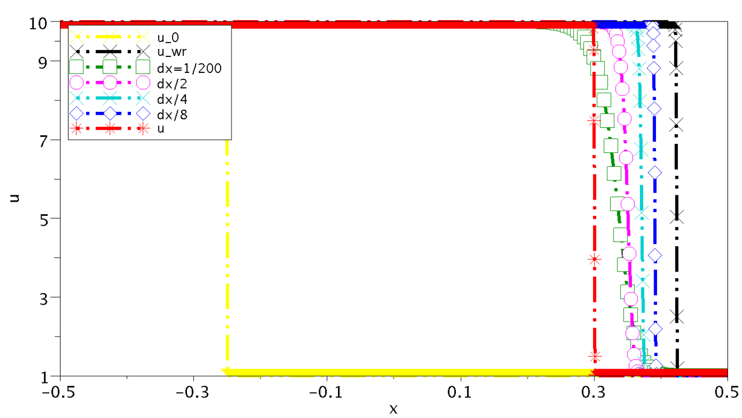


Figure 4: Upwind Scheme for (6) with non conservative diffusion term, $\alpha = 2$.

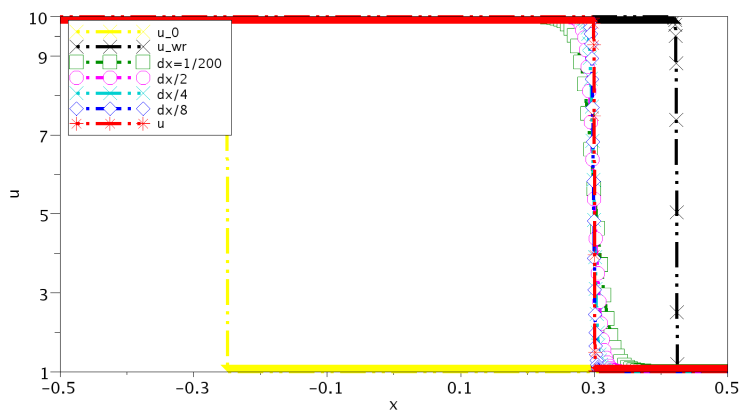


Figure 5: Centered Scheme for (6) with non conservative diffusion term, $\alpha = 1$.

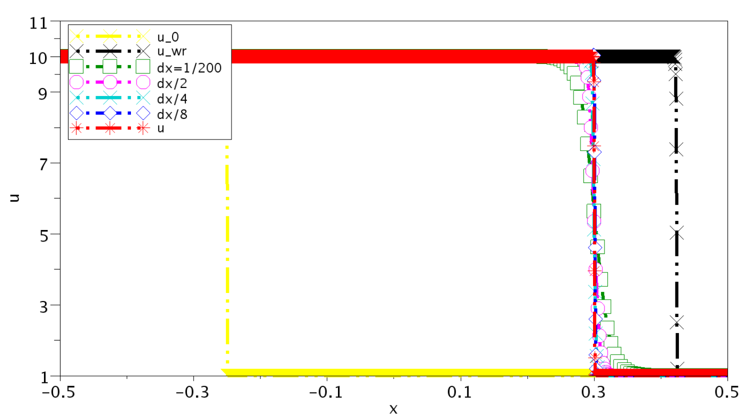


Figure 6: Centered Scheme for (6) with non conservative diffusion term, $\alpha = 1.5$.

Let us then study the following finite volume centred scheme for Equation (7), which reads:

$$\big(u_i^{(n)}\big)^2 = \big(u_i^{(n-1)}\big)^2 + \frac{4k}{3h}\Big[\big(\frac{u_{i-1}^{(n-1)} + u_i^{(n-1)}}{2}\big)^3 - \big(\frac{u_i^{(n-1)} + u_{i+1}^{(n-1)}}{2}\big)^3\Big] + \frac{k}{h^2}\,\epsilon_0\, h^\alpha\, u_i^{(n-1)}\,\Big[u_{i-1}^{(n-1)} - 2u_i^{(n-1)} + u_{i+1}^{(n-1)}\Big]. \quad (16)$$

Results for $\alpha = 1$, $\alpha = 1.5$ and $\alpha = 2$ (and ϵ_0 such that $\epsilon_0 h^\alpha = 0.2$ for $N = 200$, whatever α may be) are reported on Figures 5, 6 and 7, respectively. The numerical solution now seems to converge to the solution of (7), at least for $\alpha \in (0, 2)$. For the finest mesh and $\alpha = 2$, the diffusion is no longer sufficient to prevent some spurious oscillations near the shock. Last but not least, the additional diffusion which is necessary to recover the right shock location is considerably reduced with respect to the upwind scheme (even if the scheme still appears more diffusive than the standard upwind scheme applied to (4)), which is encouraging in view of practical extensions to Euler equations.

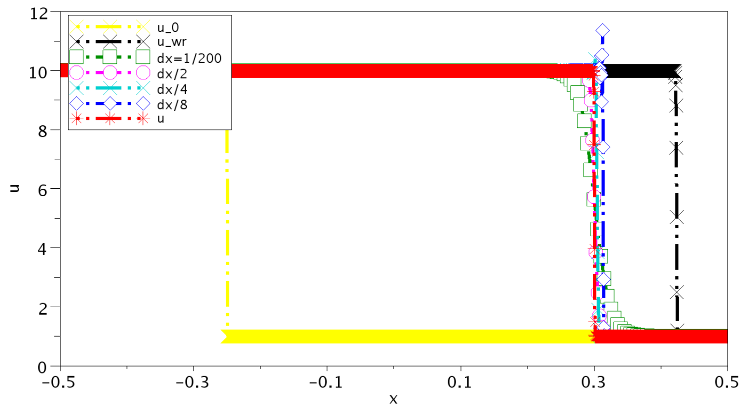


Figure 7: Centered Scheme for (6) with non conservative diffusion term, $\alpha = 2$.

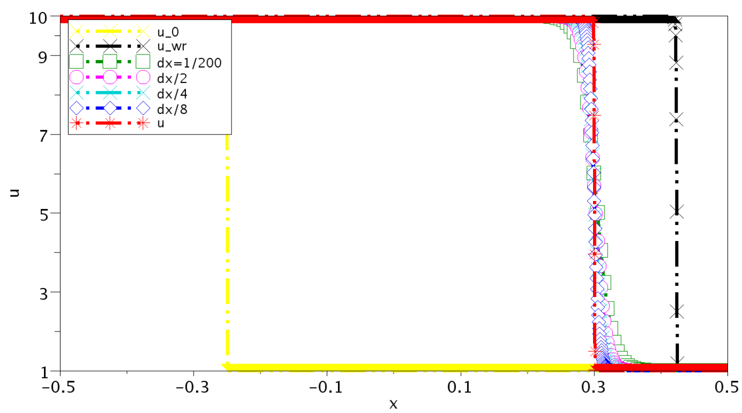


Figure 8: Modified centered Scheme for (6) with non conservative diffusion term, $\alpha = 0.5$.

Finally, we study another finite volume centred scheme for Equation (7), which reads:

$$\big(u_i^{(n)}\big)^2 = \big(u_i^{(n-1)}\big)^2 + \frac{4k}{3h}\Big[\big(\frac{(u_{i-1}^{(n-1)})^2 + (u_i^{(n-1)})^2}{2}\big)^{3/2} - \big(\frac{(u_i^{(n-1)})^2 + (u_{i+1}^{(n-1)})^2}{2}\big)^{3/2}\Big] + \frac{k}{h^2}\,\epsilon_0\,h^\alpha\,u_i^{(n-1)}\Big[u_{i-1}^{(n-1)} - 2u_i^{(n-1)} + u_{i+1}^{(n-1)}\big]. \quad (17)$$

This scheme is indeed a natural centred scheme of Equation (14) for the variable $v = u^2$. Results for $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.5$ and $\alpha = 2$ are reported on Figures 8, 9, 10 and 11 respectively. They now look quite different, since the scheme does not yields the good shock location for $\alpha = 1.5$ (and, of course, $\alpha = 2$).

4 Conclusion

We tested two discretizations for the modified equation (6):

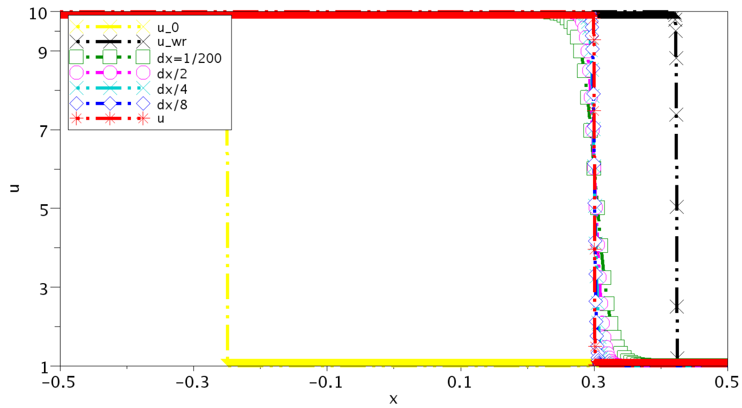


Figure 9: Modified centered Scheme for (6) with non conservative diffusion term, $\alpha = 1$.

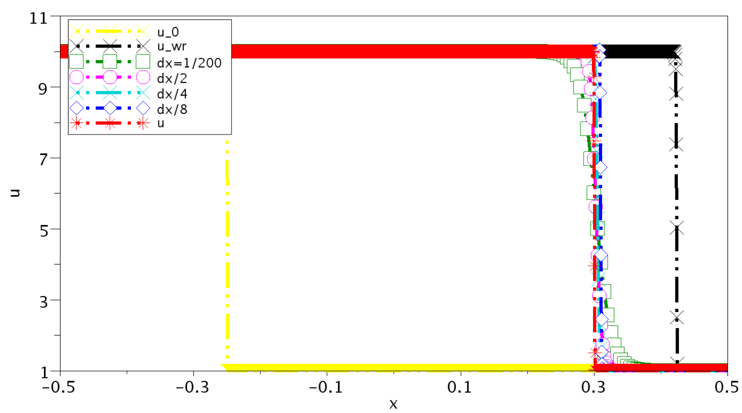


Figure 10: Modified centered Scheme for (6) with non conservative diffusion term, $\alpha = 1.5$.

– an upwind scheme for which the solution converges to the weak solution of (4) if the viscous term is predominant with respect to the numerical diffusion, that is if $\epsilon = \epsilon_0 h^\alpha$, with $\epsilon_0 > 0$ and $\alpha \in (0, 1)$.

– two centred schemes which yield less smeared solutions, with the correct shock locations for $\alpha \in (0, 1]$.

The extension of this work to Euler equations is under way, and results are encouraging. Indeed, it seems that we are able to build convergent schemes, even in the presence of shocks, using either the entropy or internal energy balance. A next step might be to use a nonlinear viscosity to avoid an excessive smearing of the solutions, following the ideas developed in [3].

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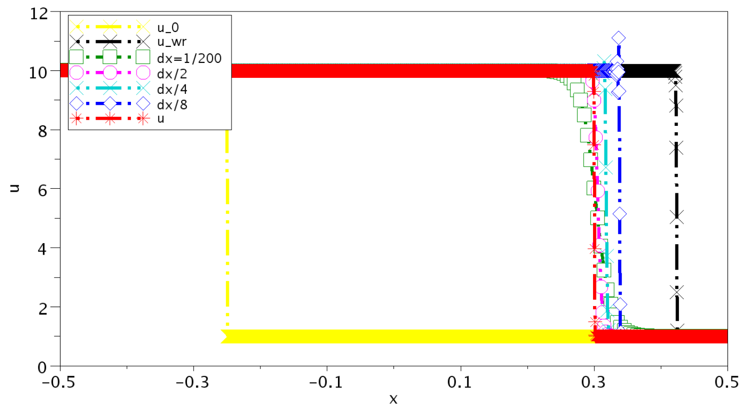

Figure 11: Modified centered Scheme for (6) with non conservative diffusion term, $\alpha = 2$.

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